Initial guesses for the **a** vector are shown in Table 1. The converged values are also shown in Table 1. The optimal control, shown in Ref. 2, and the control obtained here are shown in Fig. 1.

Approximately 25 integrations of the ϕ matrix were required to produce this trajectory. The iterations were stopped since this trajectory has $||M|| \leq 10^{-5}$ and although G is still decreasing on every iteration, it is decreasing very slowly.

The second example considered is a singular arc problem solved by Powers.[3] Again the quantity to be minimized is the final time. The state equations are the same as those shown in Eqs. (15) except that one more equation is added. The control of the Saturn vehicle is the rate of change of β. Thus if the following equation is introduced

$$\dot{\beta} = \gamma \tag{18}$$

then β becomes a state variable and the control is γ. This choice of control causes the state equations to be linear in the control variable.

For the numerical computations, variables are normalized as before where the unit of length is now chosen as one earth radius. Boundary conditions then require $u_o = 0.0220159627$, $v_o = 0.853457258$, $r_o = 1.03066079$, and $\beta_o = 0.2987038$. At the final time $u_f = 0.0$, $v_f = 0.9853923$, and $r_f = 1.02987038$. It is also required that $-K \leq \gamma \leq K$. This constraint on the control is easily satisfied by assuming the control to be of the form

$$\gamma = K \sin U(\mathbf{a},\Upsilon) \tag{19}$$

where U is a polynomial.

Initially U was specified to be a fourth order polynomial just as before. After a few iterations were made, however, it was obvious that this would not be sufficient to produce converged trajectories. The rows of the ϕ matrix were essentially linearly dependent. This resulted in very poor accuracy for the corrections in the **a** vector. This problem was easily solved by using Chebyshev polynomials normalized over the interval zero to one instead of a standard polynomial. The use of Chebyshev polynomials kept the rows from becoming linearly dependent.

For this case, initial and converged values for **a** are shown in Table 2. The control calculated here is compared with the control programs calculated by Powers in Fig. 2. Only 9 iterations were required to calculate the trajectory.

Due to the normalization, w was always set equal to one. By increasing w it is possible to produce trajectories with a slightly lower performance index than the one shown. It is also expected that a higher order polynomial could be used to produce a trajectory with a smaller performance index. The purpose of the note, however, is to show that it is relatively easy to guess a polynomial which can be used to produce reasonably good results. Thus the questions concerning the choice of best order for the polynomial and best weighting for convergence are not discussed.

Results

The possibility of guessing a functional form for a control variable is shown to be a reasonable method for calculating good suboptimal control laws. By guessing a functional form, it is necessary to calculate corrections to only a few variables rather than the entire control history. This makes the numerical algorithm used to solve optimal control problems very simple.

The results for the singular arc problem are particularly encouraging. The method used is a first order method and it converges rapidly for this problem. By experimenting with the order of the polynomial chosen and selecting the proper weighting constant, it should be possible to produce very good suboptimal control laws for this problem.

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Vibration of Slightly Curved Beams of Transversely Isotropic Composite Materials

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Material systems such as fiber reinforced plastic composites and pyrolytic graphite type materials are finding increased use in structural applications. Many of these materials are transversely isotropic,[1,2] for which the ratio of inplane modulus of elasticity to shear modulus E/G is large, with values ranging from 20 to 50. For such values of E/G, results of Ref. 1 indicate that even in relatively thin beams and plates, the effects of transverse shear deformation are important in reducing the natural frequency of flexural vibration. In the present Note, slightly curved transversely isotropic beams are analyzed, and simple approximate analytical results are derived which indicate how even very slight curvature tends to increase sharply the natural frequency.

The fundamental equations to be derived and used are those of a slightly curved Timoshenko beam. For beams of small curvature, the equations of motion can be taken as follows, with the notation of Fig. 1:

$$M' + V = \mu\ddot{\phi} \tag{1a}$$

$$N' = \rho\ddot{u} \tag{1b}$$

$$V' + (w_0'N)' = \rho\ddot{w} \tag{1c}$$

Similar equations were used in Ref. 3 but without the $\mu\ddot{\phi}$ term. Here, $w_0(x)$ is the initial shape of the beam axis, $\phi(x)$ is the rotation of cross sections, and $w'(x)$ is the rotation of tangents to the beam axis. Also, $\mu = \bar{\gamma}I$ where $\bar{\gamma}$ is a volume density; ρ is a linear density so that $\rho = \bar{\gamma}A$. The beam cross-sectional area and moment of inertia are given by A and I.

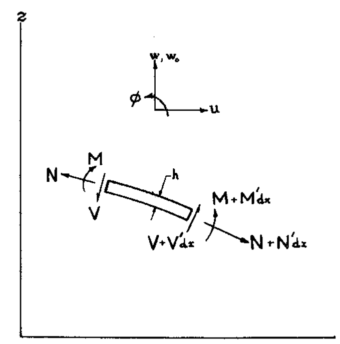

Fig. 1 Stress resultants N,V, moment M, rotation ϕ, and displacements w,u, on beam element.

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Primes and dots denote differentiation with respect to x and t, respectively.

Displacement in the x and z directions (Fig. 1) are taken as $u - z\phi$ and w, respectively, where z is measured from the beam axis. The transverse shear strain in present notation is then $\gamma_{xz} = -\phi + w'$. Appropriate constitutive relations are

$$M = EI\phi' \tag{2a}$$

$$N = EA(u' + w_0'w') \tag{2b}$$

$$V = k'AG(-\phi + w') \tag{2c}$$

where the constant k' depends on the beam cross section and equals $\pi^2/12$ for rectangular sections. Using Eqs. (2a) and (2c), Eqs. (1a) and (1c) become

$$(EI\phi')' + k'AG(-\phi + w') - \mu\ddot{\phi} = 0 \tag{3}$$

$$[k'AG(-\phi + w')]' + (w_0'N)' - \rho\ddot{w} = 0 \tag{4}$$

Equations (3) and (4) are the familiar Timoshenko beam equations which now contain the additional initial curvature term $(w_0'N)'$. In the case of shallow arches longitudinal inertia can be neglected (see Ref. 3), and setting $\ddot{u} = 0$ in Eq. (1b) yields the result that the stress resultant N is a constant. Furthermore, for the class of problems where the beam ends are fixed so that u and w are zero at $x = \pm L$ (beam span $= 2L$), Eq. (2b) can be integrated to yield an expression for the constant N

$$N\int_{-L}^{L} \frac{dx}{EA} = \int_{-L}^{L} w_0'w'dx = -\int_{-L}^{L} w_0''wdx \tag{5}$$

where

$$\int_{-L}^{L} u'dx = 0$$

has been used. The second equality in Eq. (5) was obtained by integrating by parts and noting that $w(\pm L) = 0$. For uniform beams, Eqs. (3) and (4) become

$$EI\phi'' + k'AG(-\phi + w') - \mu\ddot{\phi} = 0 \tag{6}$$

$$k'AG(-\phi + w')' - \frac{EA}{2L} w_0'' \int_{-L}^{L} w_0''wdx - \rho\ddot{w} = 0 \tag{7}$$

The second term in Eq. (7) gives the initial curvature contribution. For instance if the initial beam shape w_0 is an even function in $(-L,L)$, then subsequent modes of deflection which are odd functions in $(-L,L)$ will result in no curvature contributions.

The nondimensional quantities y_0, W, Φ, and ξ are defined by

$$w_0 = Hy_0,\ (w,\phi) = [hW,(h/L)\Phi] \sin\omega t,\ \xi = x/L \tag{8}$$

where H is the maximum value of w_0, ω is the frequency, and from here on primes will denote ξ differentiation. The resulting equations can be written as follows:

$$\Phi'' + (1/\beta)(W' - \Phi) + \bar{\mu}\Omega^2\Phi = 0 \tag{9}$$

$$\frac{1}{\beta}(W' - \Phi)' - y_0''\frac{\delta}{2}\int_{-1}^{1} y_0''Wd\xi + \Omega^2 W = 0 \tag{10}$$

where $r^2 = I/A =$ cross-sectional radius of gyration, $\beta = Er^2/k'GL^2 =$ transverse shear parameter, $\Omega^2 = \rho\omega^2L^4/EI =$ frequency parameter, $\bar{\mu} = \mu/\rho L^2 = r^2/L^2 =$ rotatory inertia parameter, $\delta = H^2/r^2 =$ curvature parameter. For slender beams (say $r/L = \frac{1}{10}$), the parameter β can take on values of order unity when $E/G = 50$, but note that $\bar{\mu}$ is still small. Rotatory inertia, therefore still has a very small effect, except of course for high values of Ω (i.e., high frequencies).

Equations (9) and (10) can be used to derive a frequency equation for flexural vibration in the case of a simply supported beam by using Fourier series expansions.[8] The initial curvature y_0'', and the solutions for W and Φ are expressed as

$$(y_0'',W) = \sum_{n=0}^{\infty} [(A_n,a_n) \cos\theta_{2n+1}\xi + (B_n,b_n) \sin\theta_{2n}\xi] \tag{11}$$

$$\Phi = \sum_{n=0}^{\infty} [c_n \sin\theta_{2n+1}\xi + d_n \cos\theta_{2n}\xi] \tag{12}$$

where $\theta_n = n\pi/2$. It is convenient to eliminate $(W' - \Phi)'/\beta$ in Eq. (10), using Eq. (9) so that Eq. (10) becomes

$$-\Phi''' - \bar{\mu}\Omega^2\Phi' - y_0''\frac{\delta}{2}\int_{-1}^{1} y_0''Wd\xi + \Omega^2 W = 0 \tag{13}$$

Equations (11) and (12) are then substituted into Eqs. (9) and (13). The usual Fourier analysis is performed and orthogonality is used to obtain the following four sets of equations for the unknown fourier coefficients $a_n,b_n,c_n,d_n (n = 0,1,2, \ldots)$:

$$(\beta\bar{\mu}\Omega^2 - 1 - \beta\theta_{2n+1}^2)c_n = \theta_{2n+1}a_n \tag{14}$$

$$(\beta\bar{\mu}\Omega^2 - 1 - \beta\theta_{2n}^2)d_n = -\theta_{2n}b_n \tag{15}$$

$$(-\bar{\mu}\Omega^2\theta_{2n+1} + \theta_{2n+1}^3)c_n + \Omega^2 a_n - A_n\frac{\delta}{2}\sum_{m=0}^{\infty}(a_mA_m + b_mB_m) = 0 \tag{16}$$

$$(\bar{\mu}\Omega^2\theta_{2n} - \theta_{2n}^3)d_n + \Omega^2 b_n - B_n\frac{\delta}{2}\sum_{m=0}^{\infty}(a_mA_m + b_mB_m) = 0 \tag{17}$$

Next, c_n and d_n are expressed in terms of a_n and b_n using Eqs. (14) and (15), so that Eqs. (16) and (17) can be written as

$$K_na_n - A_n(\delta/2)S = 0 \tag{18}$$

$$G_nb_n - B_n(\delta/2)S = 0 \tag{19}$$

where

$$K_n = \Omega^2 - (\bar{\mu}\Omega^2 - \theta_{2n+1}^2)\theta_{2n+1}^2/(\beta\bar{\mu}\Omega^2 - 1 - \beta\theta_{2n+1}^2) \tag{20}$$

$$G_n = \Omega^2 - (\bar{\mu}\Omega^2 - \theta_{2n}^2)\theta_{2n}^2/(\beta\bar{\mu}\Omega^2 - 1 - \beta\theta_{2n}^2) \tag{21}$$

$$S = \sum_{m=0}^{\infty}(a_mA_m + b_mB_m) = \int_{-1}^{1} y_0''Wd\xi \tag{22}$$

If Eqs. (18) and (19) are multiplied by A_n and B_n, respectively, added and summed over n, the result can be expressed by the following equation:

$$\left[1 - \frac{\delta}{2}\sum_{n=0}^{\infty}\left(\frac{A_n^2}{K_n} + \frac{B_n^2}{G_n}\right)\right]S = 0 \tag{23}$$

From Eq. (22), S cannot vanish if curvature is to affect the problem. Therefore the bracketted term in Eq. (23) must vanish, and yields the frequency equation for those frequencies affected by curvature.

An especially simple result can be derived when the initial shape of the beam centerline is given by the symmetric function $y_0 = \cos(\pi\xi/2)$, so that $y_0'' = -(\pi/2)^2 \cos(\pi\xi/2)$. Since the initial curvature, y_0'', is expressed as the series given in Eq. (11), it is seen that for the initial shape under discussion, $B_n = 0$ for all n, $A_n = 0$ for $n > 0$, and $A_0 = -\theta_1^2$ where $\theta_1 = -\pi/2$. Therefore the series in Eq. (23) reduces identically to one term, so that the frequency equation is $1 - \theta_1^4\delta/2K_0 = 0$, which, on substituting for K_0 becomes,

$$\Omega^2 - \theta_1^2(\theta_1^2 - \bar{\mu}\Omega^2)/[1 + \beta(\theta_1^2 - \bar{\mu}\Omega^2)] - (\delta/2)\theta_1^4 = 0 \tag{24}$$

This result can also be obtained directly by noting that the mode of vibation which is affected by curvature in the case where $y_0 = \cos(\pi\xi/2)$, is the symmetric mode given by $W = a_0 \cos(\pi\xi/2)$, $\Phi = c_0 \sin(\pi\xi/2)$. Introduction of these functions into Eqs. (9) and (10) will yield the same frequency equation as Eq. (24).

As pointed out in Ref. 1, the effect of rotatory inertia is small and can be neglected in primarily flexural vibration so that on setting $\bar{\mu} = 0$ in Eq. (24) yields,

$$\Omega^2 = \Omega_0^2 = \theta_1^4[1/(1 + \beta\theta_1^2) + (\delta/2)] \qquad (25)$$

The corresponding result for the straight Bernoulli-Euler beam is $\Omega_0^2 = \theta_1^4$. The first term in Eq. (25) corresponds to Eq. (27) of Ref. 1, where, in the present paper the total span is taken as $2L$. Note that the curvature correction in Eq. (25) is independent of the transverse shear correction and the beam span. A perturbation analysis involving Eq. (24), where $\bar{\mu}$ is regarded as a small parameter gives a first-order correction for rotatory inertia as follows:

$$\Omega^2 = \Omega_0^2[1 - \bar{\mu}\theta_1^2/(1 + \beta\theta_1^2)^2] + 0(\bar{\mu}^2) \qquad (26)$$

where Ω_0^2 is given by Eq. (25). Equation (25) also shows that initial curvature becomes important when $\delta = H^2/r^2 = 0(1)$, or when the rise H, is of the order of the thickness h, of the beam. For a rectangular cross section ($r^2 = h^2/12$) beam, take $E/G = 50$, $h/2L = \frac{1}{10}$, and $H/h = \frac{1}{4}$, so that Eq. (25) yields the result $\Omega^2 = \theta_1^4(\frac{2}{3} + \frac{3}{8}) = \theta_1^4(25/24)$, which demonstrates that even the very slight rise of $H/h = \frac{1}{4}$ is sufficient to cancel the transverse shear effect. In this example $\bar{\mu} = h^2/12L^2 = 0.0033$ which is indeed small compared to unity, so that neglect of rotatory inertia was justified.

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Generalized Finite-Element Method for Compressible Viscous Flow

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In Ref. 1, Oden derived the generalized finite-element analogue of the Navier-Stokes equations without a variational principle from the energy concept, neglecting thermal effects. Hence he is short of a finite-element analogue for the energy equation in his method. It is the purpose of this Note to derive the generalized finite-element analogue of the Navier-Stokes equations and the energy equation from the residual point of view, while continuity and the equation of state will be omitted for brevity.

Consider the Navier-Stokes equation in Cartesian tensor; i.e.,

$$\rho\left(\frac{\partial w_i}{\partial t} + w_j \frac{\partial w_i}{\partial x_j}\right) - \left(F_i + \frac{\partial p_{ij}}{\partial x_j}\right) = 0 \qquad (1)$$

Following Oden, express every variable in terms of its nodal value of an element dv. For instance,

$$\rho = \rho_M\Psi_M(\mathbf{x}), \quad w_i = w_{Qi}\Psi_Q(\mathbf{x}), \quad R = R_{Si}\Psi_S(\mathbf{x}) \qquad (2)$$

where R is the residual and Ψ_M,Ψ_Q,Ψ_S may be the same "interpolation function" $\Psi(\mathbf{x})$. Summation is implied for every repeated subscript or superscript in each term. M, Q, S, etc., refer to the nodes, while i,j refer to the vector components.

Integrating Eq. (1) with respect to dv of an element, $v^{(e)}$, with weighting function $\Psi_N(\mathbf{x})$, one obtains for the Nth node

$$\rho_M w_{Qi} a_{MQN} + \rho_M w_{Rj} w_{Qi} b_{ooj o}{}^{MRQN} - f_{Ni} + \int_{v^{(e)}} p_{ij} \frac{\partial \Psi_N}{\partial x_j}\, dv = a_{SN}R_{Si} = 0 \qquad (3)$$

where variables are nodal values

$$a_{MQN} = \int_{v^{(e)}} \Psi_M(\mathbf{x})\Psi_Q(\mathbf{x})\Psi_N(\mathbf{x})dv \qquad (4a)$$

$$b_{oojo}{}^{MRQN} = \int_{v^{(e)}} \Psi_M(\mathbf{x})\Psi_R(\mathbf{x})\Psi_{Q,j}(\mathbf{x})\Psi_N(\mathbf{x})d\mathbf{x} \qquad (4b)$$

$$f_{Ni} = \int_{v^{(e)}} F_i\Psi_N(\mathbf{x})dv + \int_{\partial v^{(e)}} p_{ij}n_j\Psi_N(\mathbf{x})dS \qquad (5)$$

$$p_{ik} = -p\delta_{ik} - \frac{2}{3}\,\mu\,\frac{\partial w_j}{\partial x_j}\,\delta_{ik} + \mu\left(\frac{\partial w_i}{\partial x_k} + \frac{\partial w_k}{\partial x_i}\right) \qquad (6)$$

For illustration, constant thermal physical constants will be assumed. Thus, one has

$$\int_{v^{(e)}} p_{ij}(\partial\Psi_N/\partial x_j)dv \equiv \int_{v^{(e)}} p_{ik}(\partial\Psi_N/\partial x_k)dv = -p_T\delta_{ik}b_k{}^{NT} - \tfrac{2}{3}\mu w_{Rj}b_{jk}{}^{RN}\delta_{ik} + \mu w_{Qi}b_{kk}{}^{QN} + \mu w_{Qk}b_{ik}{}^{QN} \qquad (7)$$

where

$$b_k{}^{NT} = \int_{v^{(e)}} \Psi_{N,k}\Psi_T dv, \quad b_{jk}{}^{RN} = \int_{v^{(e)}} \Psi_{R,j}\Psi_{N,k}dv \qquad (8a,b)$$

$$b_{ik}{}^{QN} = \int_{v^{(e)}} \Psi_{Q,i}\Psi_{N,k}dv \qquad (8c)$$

It is assumed that a_{SN} is nonsingular, which can be checked in each case, then the nodal values of residual are zero, as are the interpolated residual inside the element. Equation (3) is in agreement with the Navier-Stokes equation analogue derived by Oden, if Ψ_M is taken as unity and $\rho_M = \rho$ for incompressible flow.

Similarly, the energy equation neglecting radiation is (Ref. 2)

$$\frac{\partial Q}{\partial t} + \Phi + \frac{\partial}{\partial x_j}\left(\kappa\,\frac{\partial T}{\partial x_j}\right) - \rho\left(\frac{\partial e}{\partial t} + w_j\,\frac{\partial e}{\partial x_j}\right) + p\,\frac{\partial w_j}{\partial x_j} = 0 \qquad (9)$$

where

$$\Phi = \sigma_{ij}'\partial w_i/\partial x_j, \ \sigma_{ij}' = 2\mu e_{ij} + \lambda\partial w_j/\partial x_j \qquad (10a,b)$$

$$e_{ij} = \tfrac{1}{2}[(\partial w_i/\partial x_j) + (\partial w_j/\partial x_i)], \ \lambda = -\tfrac{2}{3}\mu \qquad (10c,d)$$

$$de = C\,dt, \ C = C_v \qquad (11a,b)$$

Integrating Eq. (9) with the weighting function $\Psi_R(\mathbf{x})$ over the element $v^{(e)}$, one finds

$$a_{HR}\dot{Q}_H + \mu(b_{jjo}{}^{LPR}W_{Li}W_{Pi} + b_{ijo}{}^{MPR}W_{Mj}W_{Pi}) + \lambda b_{jio}{}^{GPR}W_{Gj}W_{Pi} + \int_{\partial v^{(e)}}\Psi_R K(\partial T/\partial x_j)n_j dS - b_{ii}{}^{NR}KT_N - \rho_U CT_N a_{NUR} - \rho_U T_N W_{Pj} b_{ojoo}{}^{UNPR} + p_v W_{Pj} b_{ojo}{}^{VPR} = a_{SR}R_{Si} = 0 \qquad (12)$$

where b's and a's are defined analogous to Eqs. (4a) and (4b), respectively.

Including the analogues of the equation of state and the continuity equation does not complete the formulation for the generalized finite-element method, since there is no variational principle and there are more nodes than elements, in general. But there are more integrated equations than there are unknowns; the equations are applied to the boundary

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